

Splines

The term spline is derived from a flexible strip of metal commonly used by draftsmen to assist in drawing curved lines. The most commonly used splines are **cubic spline**, i.e., of order 3—in particular, cubic B-spline and cubic Bezier spline.

Splines

- To build up more complex curves, we can piece together different Bezier curves to make “splines”
- For example, we can get:
 - Positional (C^0) Continuity:
$$Q_n(0) = Q_{n-1}(1), \quad Q_n(1) = Q_{n+1}(0)$$
 - Derivative (C^1) Continuity:
$$Q_n(0) = Q_{n-1}(1), \quad Q_n(1) = Q_{n+1}(0)$$
$$Q'_n(0) = Q'_{n-1}(1), \quad Q'_n(1) = Q'_{n+1}(0)$$
- Q: How would you build an interactive system to satisfy these conditions?

Advantages of Splines



- Advantages of splines over higher-order Bezier curves:
 - Numerically more stable
 - Easier to compute
 - Fewer bumps and wiggles

Tangent (G^1) Continuity

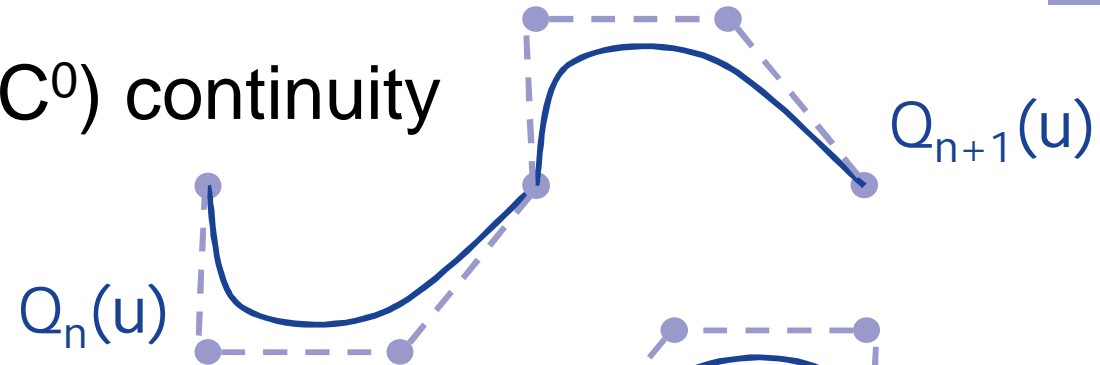
- Q: Suppose the tangents were in opposite directions but not of same magnitude – how does the curves appear?

$$\begin{aligned}Q_n(0) &= Q_{n-1}(1), & Q_n(1) &= Q_{n+1}(0) \\ Q'_n(0) &= k_1 Q'_{n-1}(1), & Q'_n(1) &= k_2 Q'_{n+1}(0)\end{aligned}$$

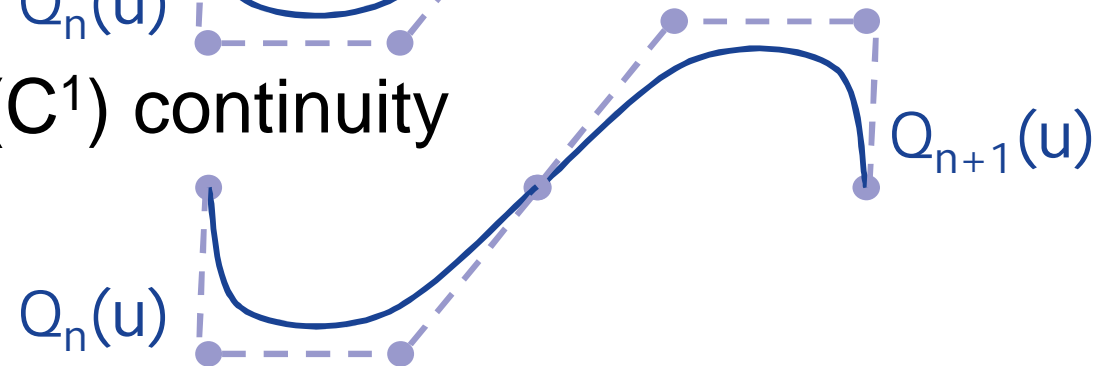
- This construction gives “tangent (G^1) continuity”
- Q: How is G^1 continuity different from C^1 ?

Explanation

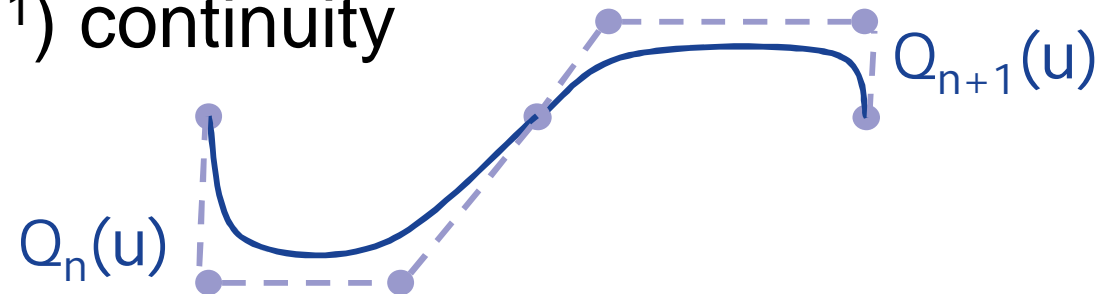
- Positional (C^0) continuity



- Derivative (C^1) continuity



- Tangent (G^1) continuity



Curvature (C^2) Continuity

- Q: Suppose you want even higher degrees of continuity – e.g., not just slopes but curvatures – what additional geometric constraints are imposed?

$$Q_n(0) = Q_{n-1}(1), \quad Q_n(1) = Q_{n+1}(0)$$

$$Q'_n(0) = Q'_{n-1}(1), \quad Q'_n(1) = Q'_{n+1}(0)$$

$$Q''_n(0) = Q''_{n-1}(1), \quad Q''_n(1) = Q''_{n+1}(0)$$

- We'll begin by developing some more mathematics.....

Operator Calculus

- Let's use a tool known as “operator calculus”
- Define the operator D by:

$$DV_i \equiv V_{i+1}$$

- Rewriting our explicit formulation in this notation gives:

$$\begin{aligned} Q(u) &= \sum_{i=0}^n \binom{n}{i} u^i (1-u)^{n-i} V_i \\ &= \sum_{i=0}^n \binom{n}{i} u^i (1-u)^{n-i} D_i V_0 = \sum_{i=0}^n \binom{n}{i} (uD)^i (1-u)^{n-i} V_0 \end{aligned}$$

- Applying the binomial theorem gives:

$$Q(u) = (uD + (1-u))^n V_0$$

Taking the Derivative

- One advantage of this form is that now we can take the derivative:

$$Q'(u) = n(uD + (1-u))^{n-1} (D-1)V_0$$

- What's $(D-1)V_0$?
- Plugging in and expanding:

$$Q'(u) = n \sum_{i=0}^{n-1} \binom{n-1}{i} u^i (1-u)^{n-1-i} D_i(V_1 - V_0)$$

- This gives us a general expression for the derivative $Q'(u)$

Specializing to $n=3$

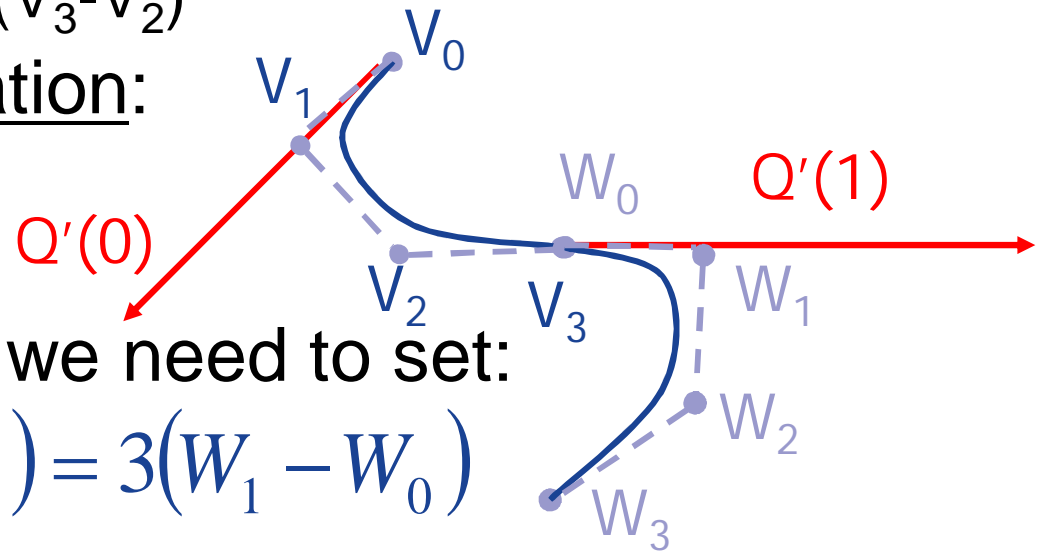
- What's the derivative $Q'(u)$ for a cubic Bezier curve?

$$Q'(u) = 3(uD + (1-u)) \cdot (D-1)V_0 \begin{cases} Q'(0) = 3(D-1)V_0 \\ Q'(1) = 3D^2(D-1)V_0 \end{cases}$$

- Note that:

- When $u=0$: $Q'(u) = 3(V_1 - V_0)$
- When $u=1$: $Q'(u) = 3(V_3 - V_2)$

- Geometric interpretation:



- So for C^1 continuity, we need to set:

$$3(V_3 - V_2) = 3(W_1 - W_0)$$

Taking the Second Derivative

- Taking the derivative once again yields:

$$Q''(u) = n(n-1)(uD + (1-u))^{n-2} (D-1)^2 V_0$$

- What does $(D-1)^2$ do?

Second-Order Continuity

- So the conditions for second-order continuity are:

$$(V_3 - V_2) = (W_1 - W_0)$$

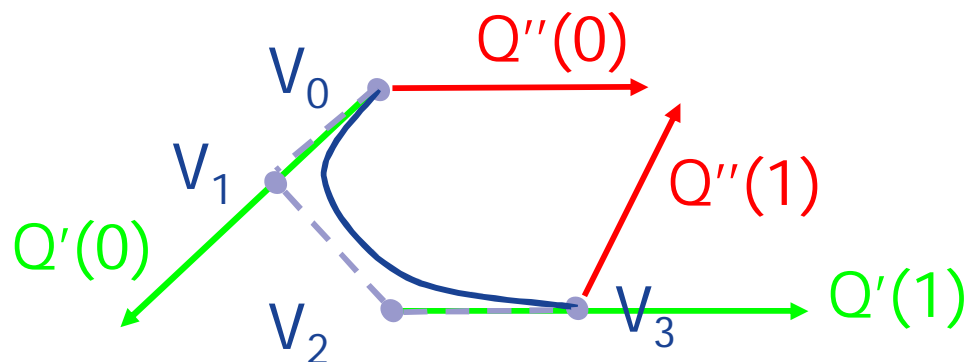
$$(V_3 - V_2) - (V_2 - V_1) = (W_2 - W_1) - (W_1 - W_0)$$

- Putting these together gives: $W_0 = V_3$

$$W_1 = (V_3 - V_2) + W_0 = 2V_3 - V_2$$

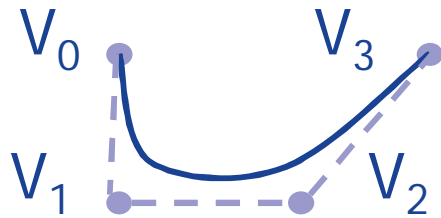
$$W_2 = 2(V_3 - V_2) - (V_2 - V_1) + W_1 = V_1 - 4V_2 + 4V_3$$

- Geometric interpretation



C^3 Continuity

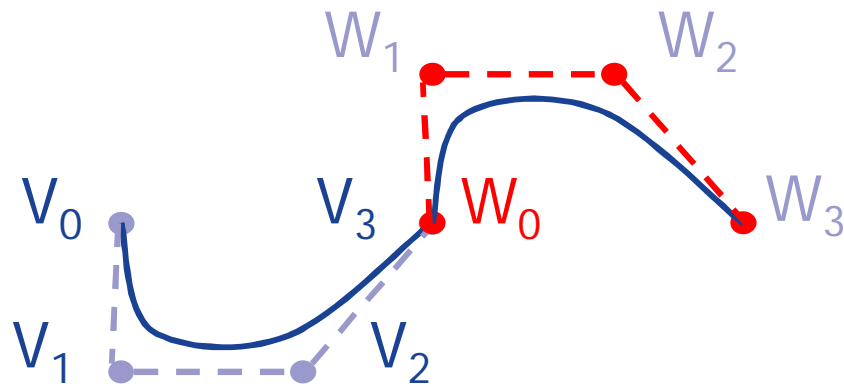
$$\underline{Q'''(u) = n(n-1)(n-2)(uD + (1-u))^{n-3} (D-1)^3 V_0}$$



C^3 Continuity

$$Q'''(u) = n(n-1)(n-2)(uD + (1-u))^{n-3} (D-1)^3 V_0$$

$$W_0 = V_3$$

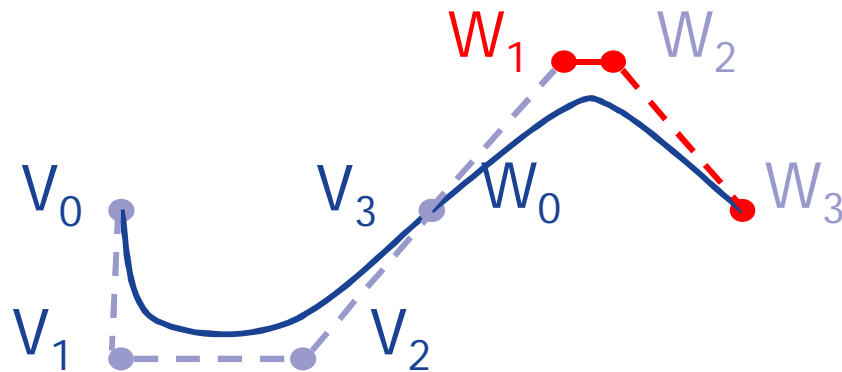


C^3 Continuity

$$Q'''(u) = n(n-1)(n-2)(uD + (1-u))^{n-3} (D-1)^3 V_0$$

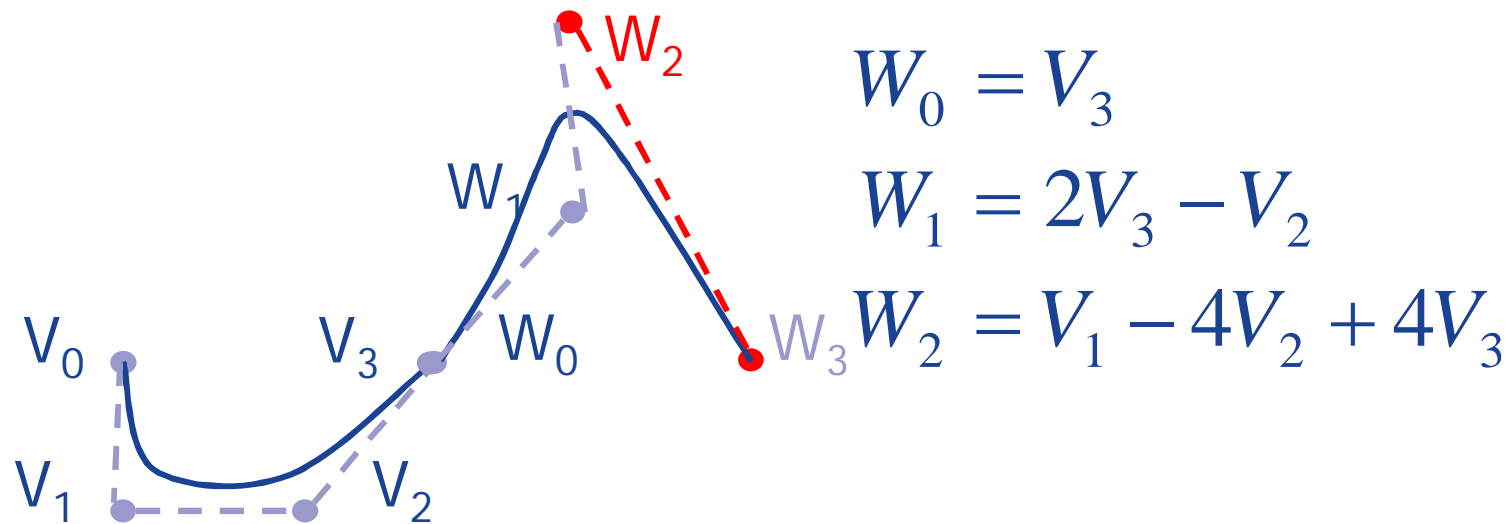
$$W_0 = V_3$$

$$W_1 = 2V_3 - V_2$$



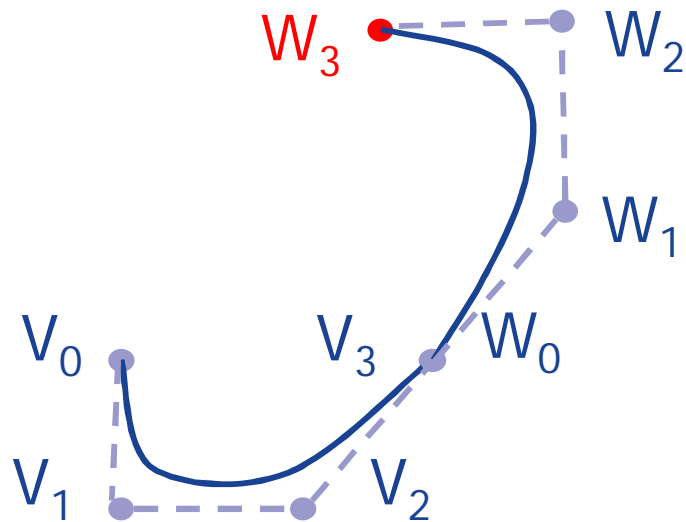
C^3 Continuity

$$Q'''(u) = n(n-1)(n-2)(uD + (1-u))^{n-3} (D-1)^3 V_0$$



C^3 Continuity

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$$W_0 = V_3$$

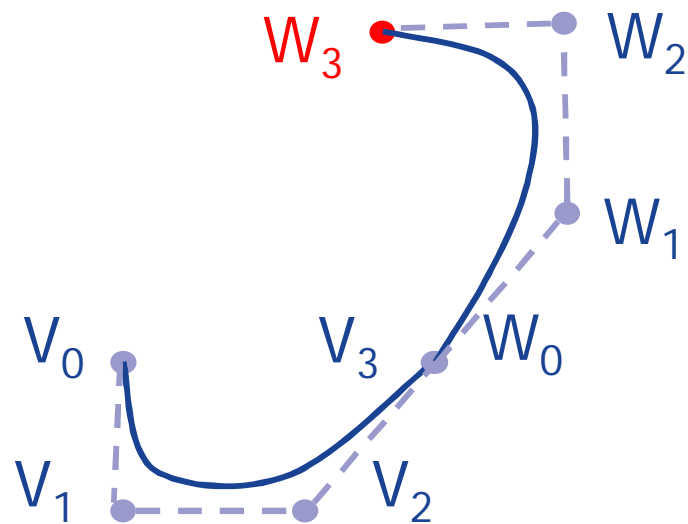
$$W_1 = 2V_3 - V_2$$

$$W_2 = V_1 - 4V_2 + 4V_3$$

$$W_3 = 8V_3 - 12V_2 + 6V_1 - V_0$$

C^3 Continuity

$$Q'''(u) = n(n-1)(n-2)(uD + (1-u))^{n-3} (D-1)^3 V_0$$



$$W_0 = V_3$$

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■ Summary of continuity conditions

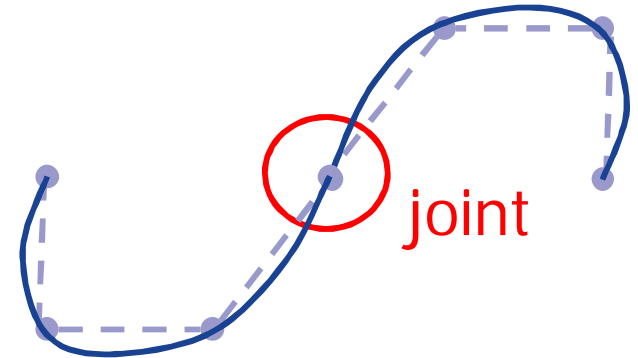
- C^0 straightforward, but generally not enough
- C^3 is too constrained (with cubics)

Creating Continuous Splines

- We'll look at three ways to specify splines with C^1 and C^2 continuity
 - C^2 interpolating splines
 - B-splines
 - Catmull-Rom splines

C^2 Interpolating Splines

- The control points specified by the user, called “joints”, are interpolated by the spline
- For each of x and y , we needed to specify 3 conditions for each cubic Bezier segment.
- So if there are m segments, we’ll need $3m-1$ conditions
- Q: How many these constraints are determined by each joint?



In-Depth Analysis

- At each interior joint j , we have:
 - Last curve ends at j
 - Next curve begins at j
 - Tangents of two curves at j are equal
 - Curvature of two curves at j are equal
- The m segments give:
 - $m-1$ interior joints
 - 3 conditions
- The 2 end joints give 2 further constraints:
 - First curve begins at first joint
 - Last curve ends at last joint
- Gives $3m-1$ constraints altogether

End Conditions

- The analysis shows that specifying $m+1$ joints for m segments leaves 2 extra degree of freedom
- These 2 extra constraints can be specified in a variety of ways:
 - An interactive system
 - Constraints specified as [user inputs](#)
 - “Natural” cubic splines
 - Second derivatives at endpoints defined to be 0
 - Maximal continuity
 - Require C^3 continuity between first and last pairs of curves

C² Interpolating Splines

- Problem: Describe an interactive system for specifying C² interpolating splines
- Solution:
 1. Let user specify first four Bezier control points
 2. This constraints next 2 control points – draw these in.
 3. User then picks 1 more
 4. Repeat steps 2-3.

Global vs. Local Control

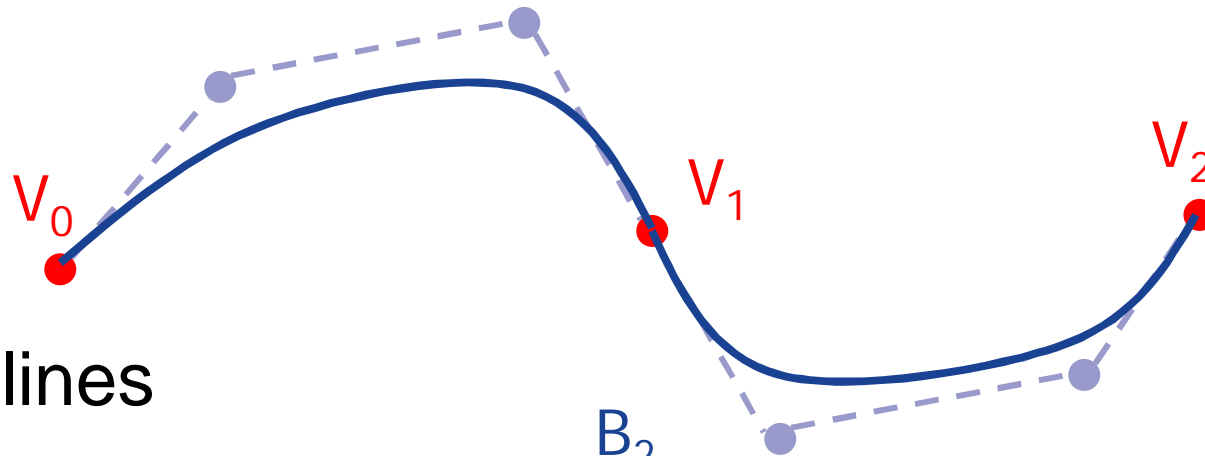
- These C^2 interpolating splines yield only “global control” – moving any one joint (or control point) changes the entire curve!
- Global control is problematic:
 - Makes splines difficult to design
 - Makes incremental display inefficient
- There’s a fix, but nothing comes for free. Two choices:
 - B-splines
 - Keep C^2 continuity
 - Give up interpolation
 - Catmull-Rom Splines
 - Keep interpolation
 - Give up C^2 continuity – provides C^1 only

B-Splines

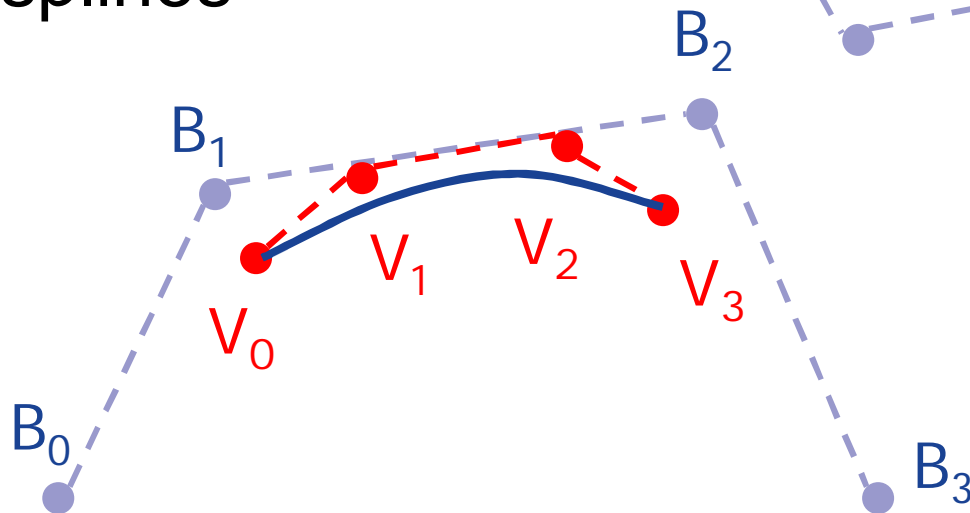
- Previous construction (C^2 interpolating splines):
 - Choose joints, constrained by the “A-frames”
- New construction (B-splines):
 - Choose points on A-frames
 - Let these determine the rest of Bezier control points and joints
- The B-splines I’ll describe are known more precisely as “uniform B-splines”

Explanation

■ C^2 interpolating spline



■ B-splines



B-Spline Construction

$$P(u) = \sum_{k=0}^n p_k B_{k,d}(u), \quad u_{\min} \leq u \leq u_{\max}, \quad 2 \leq d \leq n+1$$

$n+1$ control points

polynomials of degree $d-1$

$$B_{k,1}(u) = \begin{cases} 1, & \text{if } u_k \leq u \leq u_{k+1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{k,d}(u) = \frac{u - u_k}{u_{k+d-1} - u_k} B_{k,d-1}(u) + \frac{u_{k+d} - u}{u_{k+d} - u_{k+1}} B_{k+1,d-1}(u)$$

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- The points specified by the user in this construction are called “deBoor points”

Algebraic Construction of B-Splines

$$P(u) = \sum_{k=0}^3 B_k F_{k,3}(u)$$

$$= B_0 F_{0,3}(u) + B_1 F_{1,3}(u) + B_2 F_{2,3}(u) + B_3 F_{3,3}(u)$$

$$F_{0,3}(u) = \frac{1}{6}(1-u)^3,$$

$$F_{1,3}(u) = \frac{1}{6}(3u^2 - 6u + 4),$$

$$F_{2,3}(u) = \frac{1}{6}(-3u^3 + 3u^2 + 3u + 1),$$

$$F_{3,3}(u) = \frac{1}{6}u^3$$

- $V_1 = (1 - 1/3)B_1 + 1/3 B_2$
- $V_2 = (1 - 2/3)B_1 + 2/3 B_2$
- $V_0 = \frac{1}{2} [(1 - 2/3)B_0 + 2/3 B_1] + \frac{1}{2} [(1 - 1/3)B_1 + 1/3 B_2]$
 $= 1/6 B_0 + 2/3 B_1 + 1/6 B_2$
- $V_3 = 1/6 B_1 + 2/3 B_2 + 1/6 B_3$


B-Spline Properties

- Here are some properties of B-splines:
 - C² continuity
 - Approximating
 - Does not interpolate deBoor points
 - Locality
 - Each segment determined by 4 deBoor points
 - Each deBoor point determine 4 segments
 - Convex hull
 - Curves lies inside convex hull of deBoor points

Application



The use of splines for modeling automobile bodies seems to have several independent beginnings. Credit is claimed on behalf of de Casteljou at Citroën, Pierre Bézier at Renault, and de Boor at General Motors.



Scope of Research



Cubic splines are used to fit a smooth curve to a series of points with a piecewise series of cubic polynomial curves. In addition to their use in interpolation, they are of particular interest to engineers because the spline is defined as the shape that a thin flexible beam (of constant flexural stiffness) would take up if it was constrained to pass through the defined points.

